Products and Covering of Lattice-valued Finite Automata

Lou Quanfu, Hu Zhonggang

Nanchang Institute of Science & Technology, Nanchang, Jiangxi, 330108, China

Email:zhonggang-hu@163.com

Keywords: lattice-valued finite automata; homomorphism; full direct product; restricted product; cascade product; wreath product; coverings.

Abstract: In this paper, the new concept of covering for lattice-valued finite automata was given. The product relations between various lattice-valued finite automata, the covering relations between various products of two lattice-valued finite automata, and covering relations between products of two lattice-valued finite automata and products of the other two lattice-valued finite automata which cover them are studied.

1. Introduction

Since the fuzzy set theory was put forward in [1], the research on automata theory and code theory has also unfolded in [2] with the method of fuzzy set. After that, the research and application of fuzzy automata became more and more in-depth. [3] began to study the algebra of fuzzy automata. [4] has made a detailed study of the algebraic properties of fuzzy automata. In 2003, Li Yongming in [5] proposed the theory of Lattice-valued automata and its language, and built the words-based computational model on the theory of more extensive Lattice-valued automata. In automata theory, product is one of the basic operations, and the product and covering relation of different forms play a very important role in the decomposition of automata. The products of lattice-valued finite automata are studied in [6]. [3] studies the product and covering relation of a fuzzy finite state machine. A new covering concept is given in this paper, and the covering relation between the products of the Lattice-valued finite automata given in [6] is studied. The covering relation between the product of lattice-valued finite state machines and the product of those covering them is also discussed.

2. Basic Concepts and Symbols

Definition 2.1[7] A lattice-valued finite automata (LFA) is a five tuple, $M=(Q,X,Y,\mu,\sigma)$,in which Q,X,Y are non-empty finite state set, non-empty finite input symbol set, non-empty finite output symbol set respectively. μ is a L-fuzzy subset in $Q\times X\times Q$, that is $\mu: Q\times X\times Q\to L$, which is called fuzzy state transfer function. σ is a L-fuzzy subset in $Q\times X\times Y$,that is, $\sigma: Q\times X\times Y\to L$,which is called fuzzy output function. For a lattice-valued finite automata, the following conditions are satisfied:

$$(\forall q\in Q)(\forall x\in X)((\exists p\in Q),\mu(q,x,p)>0\Leftrightarrow(\exists y\in Y),\sigma(q,x,y)>0)$$

Because only the transfer structure of automata is discussed, the lattice-valued finite automata in the definition above is expressed as $M=(Q,X,\mu)$, which is called a finite state machine. We agree that semi-group operations "$\bullet$" here are commutative.

Definition 2.2[8] Suppose $M=(Q,X,\mu)$ is a lattice-valued finite state machine, among which $\forall x\in X^*, a\in X$. Λ is an empty character, X^* represents the set of all strings of finite length on the set X . $\mu^*: Q\times X^*\times Q\to L$ is defined as follows：

$$\mu^*(q,\Lambda,p)=\begin{cases}1, q=p\\ 0, q\neq p\end{cases},\quad \mu^*(q,xa,p)=\vee_{r\in Q}\{\mu^*(q,x,r)\bullet\mu(r,a,p)\}$$

3. The Covering of Lattice-valued Finite State Machines

Definition 3.1 Suppose $M_i=(Q_i,X_i,\mu_i)$ is a lattice-valued finite state machine, $i=1,2$. $\eta:Q_2\to Q_1$ is a partial function,and $\xi:X_1\to X_2$ is another function.Expand ξ to ξ^*, making $\xi^*(\Lambda)=\Lambda$, $\xi^*(x)=\xi(x_1)\xi(x_2)\cdots$
$\xi(x_n)$, among which $x=x_1x_2\cdots x_n, x_i\in X_1^*, i=1,2,\cdots,n$. Then (η,ξ) is called a covering of $M_2\to M_1$,
marked as $M_2\le M_1$. If satisfied: $\mu_1(\eta(q_2),x,\eta(p_2))\le\mu_2(q_2,\xi(x),p_2)$, $\forall p_2,q_2\in Q_2, x\in X_1$

Proposition 3.1 Suppose $M_i=(Q_i,X_i,\mu_i)$ is a lattice-valued finite state machine, $i=1,2$. (η,ξ) is a covering of $M_2\to M_1$, then $\mu_1^*(\eta(q_2),x,\eta(p_2))\le\mu_2^*(q_2,\xi(x),p_2)$, $\forall p_2,q_2\in Q_2, x\in X_1^*$.

Proof The induction of character length n. When $n=2$, making $x=x_1x_2, y=y_1y_2$,

$$\mu_1^*(\eta(q_2),xy,\eta(p_2))=\vee_{r\in Q}\{\mu_1(\eta(q_2),x_1,r)\wedge\mu_1(r,x_2,\eta(p_2))\}$$

$$\le\vee_{r\in Q_1}\{\mu_2(q_2,x_1,\eta(p_2))\wedge\mu_2(r,x_2,p_2)|\eta(r_1)=r\}\le\mu_2^*(q_2,\xi(x_1x_2),p_2)$$

Suppose the conclusion is set up when $n=k$, then $n=k+1$, making $x=\prod_{i=1}^{k+1}x_i, x_i\in X_1$.

$$\mu_1^*\left(\eta(q_2),\prod_{t=1}^{k+1}x_i,\eta(p_2)\right)=\vee\left\{\mu_1^*\left(\eta(q_2),\prod_{i=1}^{k}x_i,r_1\right)\wedge\mu_1(r_1,x_{k+1},\eta(p_2))|r_1\in Q_1\right\}$$

$$=\vee\left\{\mu_1^*\left(\eta(q_2),\prod_{t=1}^{k}x_i,\eta(r_2)\right)\wedge\mu_1(\eta(r_2),x_{k+1},\eta(p_2))|\eta(r_2)=r_1,r_1\in Q_1\right\}$$

$$\le\vee\left\{\mu_2^*\left(q_2,\xi\left(\prod_{i=1}^{k}x_i\right),\eta(r_2)\right)\wedge\mu_1(\eta(r_2),x_{k+1},\eta(p_2))|r_2\in Q_2\right\}$$

$$\le\vee\left\{\mu_2^*\left(q_2,\xi\left(\prod_{i=1}^{k}x_i\right),r_2\right)\wedge\mu_2(r_2,\xi(x_{k+1}),\eta(p_2))|r_2\in Q_2\right\}=\mu_2^*\left(q_2,\xi\left(\prod_{i=1}^{k+1}x_i\right),p_2\right).$$

Definition 3.2[8] Suppose $M_1=(Q_1,X_1,\mu_1)$ and $M_2=(Q_2,X_2,\mu_2)$ are two lattice-valued finite state machines. (1)A pair of mappings $(\alpha,\beta),\alpha:Q_1\to Q_2,\beta:X_1\to X_2$ are homomorphic,marked as $(\alpha,\beta):M_1\to M_2$, suppose $\mu_1(q,x,p)\le\mu_2(\alpha(q),\beta(x),\alpha(p)),\forall p,q\in Q_1,\forall x\in X_1$. (2)$(\alpha,\beta)$ is called a strong homomor-phic mapping,suppose

$$\mu_2(\alpha(q),\beta(x),\alpha(p))=\vee\{\mu_1(q,x,t)|t\in Q_1,\alpha(t)=\alpha(p)\},\forall p,q\in Q_1,\forall x\in X_1.$$

If α,β is surjective (injective), then homomorphism $(\alpha,\beta):M_1\to M_2$ is surjective (injective); If α,β is one-to-one mapping, then homomorphism (strong homomorphism) $(\alpha,\beta):M_1\to M_2$ is called isomorph-ism (strong isomorphism).

Note: If $X_1=X_2,\beta$ is an identity mapping, then simply marking $\alpha:M_1\to M_2$, then correspondently naming α as homomorphism (strong homomorphism).

Lemma 3.1[8] Suppose $M_i=(Q_i,X_i,\mu_i)$ is a lattice-valued finite state machine, $i=1,2$ .If $(\alpha,\beta):M_1\to M_2$ is a strong homomorphism, and α is an injective mapping, then

$\forall p,q\in Q_1,\forall x\in X_1$, making $\mu_2(\alpha(q),\beta(x),\alpha(p))=\mu_1(q,x,p)$.

Theorem 3.1 Suppose $M_i=(Q_i,X_i,\mu_i)$ is a lattice-valued finite state machine, $i=1,2$ .If $(\alpha,\beta):M_1\to M_2$ is a homomorphism,then(1)If this homomorphism is an epimorphism, and α is an injective mapping, then $M_2\le M_1$;(2)If α is an injective mapping, then $M_1\le M_2$.

Proof (1) $(\alpha,\beta):M_1\to M_2$ is a strong homomorphism, so there exist full functions $\alpha:Q_1\to Q_2$ and $\beta:X_1\to X_2$,making $\eta=\alpha:Q_1\to Q_2,\xi:X_2\to X_1$; As β is a full function, then $\forall x_2 \in X_2$,at least existing $x_1 \in X_1$, which makes $\beta(x_1) = x_2$, thus $\xi(x_2) = a$ .Furthermore, (α, β) is a strong homomorphism, and α is an injective mapping.Based on Lemma 3.1, which makes $\forall p, q \in Q_1, \forall x \in X_1$,thus $\mu_2(\alpha(q), \beta(x), \alpha(p)) = \mu_1(q, x, p)$ .If $\xi(x_2) = x_1$, we can conclude that $\mu_2(\alpha(q), x_2, \alpha(p)) = \mu_2(\alpha(q), \beta(x_1), \alpha(p)) = \mu_1(q, x_1, p) = \mu_1(q, \xi(x_1), p)$, for (η, ξ) is a covering of $M_1 \to M_2$, thus $M_2 \le M_1$.

(2) $(\alpha, \beta): M_1 \to M_2$ is a homomorphism, so there exist mappings $\alpha: Q_1 \to Q_2$ and $\beta: X_1 \to X_2$, making $\forall p_1, q_1 \in Q_1, \forall x_1 \in X_1$,then $\mu_1(q_1, x_1, p_1) \le \mu_2(\alpha(q_1), \beta(x_1), \alpha(p_1))$,making $\eta: Q_2 \to Q_1, \eta(q_2)$,If $\alpha(q_1) = q_2$, thus q_1 is uniquely determined because α is an injective mapping, thus η is a part of full function, making $\xi = \beta: X_1 \to X_2$,thus $\forall p_2, q_2 \in Q_2, \forall x_1 \in X_1$, Then $\mu_1(\eta(q_2), x_1, \eta(p_2)) \le \mu_2(q_2, \beta(x_1), p_2)$, Therfore, (η, ξ) is a covering of $M_2 \to M_1$, that is, $M_1 \le M_2$.

Theorem 3.2 Suppose $M_i = (Q_i, X_i, \mu_i)$ is a lattice-valued finite state machine, $i = 1, 2$. If $(\alpha, \beta): M_1 \to M_2$ is a homomorphism , then(1)If this homomorphism is an epimorphism, and α is an injective mapping, then $M_2 \le M_1$;(2)If α is an injective mapping, then $M_1 \le M_2$.

4. Products of Lattice-valued Finite Automata

Definition 4.1[7] Suppose $M_i = (Q_i, X_i, \mu_i)$ is a lattice-valued finite state machine, $i = 1, 2$, naming $M_1 \times M_2 = (Q_1 \times Q_2, X_1 \times X_2, \mu_1 \times \mu_2)$ as a full direct product of M_1 and M_2 , among which $\mu_{M_1 \times M_2}((Q_1 \times Q_2) \times (X_1 \times X_2) \times (Q_1 \times Q_2)) \to L$, $\forall (q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2, \forall (x_1, x_2) \in X_1 \times X_2$, making $\mu_{M_1 \times M_2}((q_1, q_2), (x_1, x_2), (p_1, p_2)) = \mu_{M_1}(q_1, x_1, p_1) \wedge \mu_{M_2}(q_2, x_2, p_2)$.

Definition 4.2[7] Suppose $M_i = (Q_i, X, \mu_i)$ is a lattice-valued finite state machine, $i = 1, 2$, naming $M_1 \wedge M_2 = (Q_1 \times Q_2, X, \mu_1 \wedge \mu_2)$ as a retricted direct product of M_1 and M_2 , among which

$\mu_{M_1 \wedge M_2}((Q_1 \times Q_2) \times X \times (Q_1 \times Q_2)) \to L$, $\forall (q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2, x \in X$, making

$$\mu_{M_1 \wedge M_2}((q_1, q_2), x, (p_1, p_2)) = \mu_{M_1}(q_1, x, p_1) \wedge \mu_{M_2}(q_2, x, p_2).$$

Definition 4.3[7] Suppose $M_i = (Q_i, X_i, \mu_i)$ is a lattice-valued finite state machine, $i = 1, 2$, naming $M_1 \circ M_2 = (Q_1 \times Q_2, X_1^{Q_2} \times X_2, \mu_1 \circ \mu_2)$ as a wreath product of M_1 and M_2 , among which $\mu_{M_1 \circ M_2}((Q_1 \times Q_2) \times (X_1^{Q_2} \times X_2) \times (Q_1 \times Q_2)) \to L$, $X_1^{Q_2} = \{ f \mid f: Q_2 \to X_1 \}$, $\forall ((q_1, q_2), (f, x_2), (p_1 \times p_2)) \in (Q_1 \times Q_2) \times (X_1^{Q_2} \times X_2) \times (Q_1 \times Q_2)$, making $\mu_{M_1 \circ M_2}((q_1, q_2), (f, x_2), (p_1, p_2)) = \mu_{M_1}(q_1, f(q_2), p_1) \wedge \mu_{M_2}(q_2, x_2, p_2)$.

Definition 4.4[7] Suppose $M_i = (Q_i, X_i, \mu_i)$ a lattice-valued finite state machine, $i = 1, 2$,naming $M_1 \omega M_2 = (Q_1 \times Q_2, X_2, \mu_1 \omega \mu_2)$ as a cascade product of M_1 and M_2 , among which

$\mu_{M_1 \omega M_2}((Q_1 \times Q_2) \times X_2 \times (Q_1 \times Q_2)) \to L$, $\omega: Q_2 \times X_2 \to X_1$ is a function, $\forall (q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$, $\forall x_2 \in X_2$,making $\mu_{M_1 \omega M_2}((q_1, q_2), x_2, (p_1, p_2)) = \mu_{M_1}(q_1, \omega(q_2, x_2), p_1) \wedge \mu_{M_2}(q_2, x_2, p_2)$.

Theorem 4.1 Suppose $M_i = (Q_i, X_i, \mu_i)$ is a lattice-valued finite state machine, $i = 1, 2$ .then (1) $M_1 \wedge M_2 \le M_1 \times M_2$, among which $X_1 = X_2 = X$, (2) $M_1 \omega M_2 \le M_1 \circ M_2$, (3) $M_1 \circ M_2 \le M_1 \times M_2$, (4) $M_1 \omega M_2 \le M_1 \times M_2$.

Proof (1) Define $\eta: Q_1 \times Q_2 \to Q_1 \times Q_2$ as an identity mapping of $Q_1 \times Q_2$. Obviously the conclusion is true.

(2) Define $\eta: Q_1 \times Q_2 \to Q_1 \times Q_2$ as an identity mapping of $Q_1 \times Q_2$. Obviouly η is a partial function. Define $\xi: X_2 \to X_1^{Q_2} \times X_2$, $\xi(a) = (f, a), \forall a \in X_2$,among which $f: Q_2 \to X_1, f(p_2) = a = \omega(p_2, a), \forall p_2 \in Q_2$, ξ as a function, and

$$\mu_{M_1 \omega M_2}(\eta(q_1, q_2), a, \eta(p_1, p_2)) = \mu_{M_1 \omega M_2}((q_1, q_2), a, (p_1, p_2))$$

$$= \mu_{M_1}(p_1, \omega(p_{2,} a), q_1) \wedge \mu_{M_2}(p_2, a, q_2) = \mu_{M_1}(p_1, f(p_2), q_1) \wedge \mu_{M_2}(p_2, a, q_2)$$

$= \mu_{M_1 \omega M_2}((q_1, q_2), (f, a), (p_1, p_2)) = \mu_{M_1 \omega M_2}((q_1, q_2), \xi(a), (p_1, p_2))$, for $M_1 \omega M_2 \le M_1 \circ M_2$.

(3)Define $\xi: X_1^{Q_2} \times X_2 \to X_1 \times X_2$ as $\xi(f, a) = (f(p_2), a)$, among which $f: Q_2 \to X_1, f(q_2) = a_1, \forall a \in X_2, p_2 \in Q_2$, and define η as an identity mapping of $Q_1 \times Q_2$, easy to prove $M_1 \circ M_2 \le M_1 \times M_2$.

(4) Based on(2)and(3), $M_1 \omega M_2 \le M_1 \times M_2$.

Theorem 4.2 Suppose $M_i = (Q_i, X_i, \mu_i)$ is a lattice-valued finite state machine, $i = 1, 2, 3$ .If $M_1 \le M_2$, then(1) $M_1 \times M_3 \le M_2 \times M_3, M_3 \times M_1 \le M_3 \times M_2$, and if $X_1 = X_2 = X_3 = X$, then $M_1 \wedge M_3 \le M_2 \wedge M_3$, $M_3 \wedge M_1 \le M_3 \wedge M_2$,(2)If among any $\omega_1: Q_3 \times X_3 \to X_1$, there exists $\omega_2: Q_3 \times X_3 \to X_2$, making $M_1 \omega_1 M_3 \le M_2 \omega_2 M_3$. If (η, ξ) is a covering of $M_2 \to M_1$, and ξ is a surjection,then among any $\omega_1: Q_1 \times X_1 \to X_3$, existing $\omega_2: Q_2 \times X_2 \to X_3$, making $M_3 \omega_1 M_1 \le M_3 \omega_2 M_2$, (3) $M_1 \circ M_3 \le M_2 \circ M_3, M_3 \circ M_1 \le M_3 \circ M_2$.

Acknowledgements

Science and technology research project of Jiangxi Provincial Education Department (GJJ161223).

References

[1] Zadeh L A. Fuzzy sets [J].Information and Controls, 1965, 8(3):338-353.

[2] Mo Zhiwen,Pen Jiayin.An algorithm of test for fuzzy codes[J].Appl Math J Chinese Univser B,2001,16(1):88-94.

[3] Malik D S,Mordeson J N,Sen M K.Products of fuzzy finite state machines[J].Fuzzy Sets and Systems,1997,92(2):95-102.

[4] DasP.On some properties of fuzzy semiautomata over a finite group [J].Inform Science, 1997, 101(1):71-84.

[5] Li Y M,.Lattice-valued automata and their languages[J].Journal of Shanxi Normal University(Natural Science Edition),2,2003,31(4):1-6.

[6] Liu J,Mo Z W.. Products of lattice-valued finite automata [J]. Applied Mathematics A Journal of Chinese Universities, 2209, 24 (1):121-126.

[7] Lei H X, Pan C. Lattice-valued finite automata and its properties [J]. Journal of Neijiang Teachers College (Natural Science Edition), 2006, 21(4):9-12.

[8] Lei H X. Homomorphism and quotient automata of lattice-valued finite states automata [J].Computer Engineering andApplications, 2010, 46(27):52-53.